

Induced superconductivity in 2D electronic systems

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The approach applicable for spatially inhomogeneous and time-dependent problems associated with the induced superconductivity in low dimensional electronic systems is developed. This approach is based on the Fano–Anderson model which describes the decay of a resonance state coupled to a continuum. We consider two types of junctions made of a ballistic 2D electron gas placed in a tunnel finite-length contact with a bulk superconducting leads. We calculate the spectrum of the bound states, supercurrent, and the current-voltage curve which show a rich structure due to the presence of induced gap and dimensional quantization.

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I. INTRODUCTION

Recent progress in studies of transport in graphene that followed the seminal work of Ref.¹ has boosted the interest in properties of contacts between the graphene sheets and various types of electrodes attached to them. Of special interest are contacts between graphene and superconducting electrodes due to the specific nature of the Andreev reflection in graphene². In practical devices, such type of contacts has the form of a superconducting lead placed on top of the graphene layer which partially overlaps with the lead. Following Ref.², the model most commonly used for description of such contacts assumes that the superconductor simply introduces certain pairing potential in the part of layer which is immediately under the superconductor as well as shifts its Fermi level away from the Dirac point. As a result, the contact is treated as being formed between the usual normal graphene layer and such piece of graphene where both the induced pairing potential and the high doping level are present. Various modifications of this model have been extensively used for studying transport properties of graphene contacts^{3–9}.

Though this model is a significant step forward in understanding the induced superconductivity in graphene, it still oversimplifies the proximity effects which the bulk superconductor has on the underlying normal sheet. To study these effects more carefully one can look at other models used for contacts between superconductors and low-dimensional electronic systems. For example, in Refs.^{10,11} the proximity effect in a two-dimensional (2D) electron gas was modelled by a uniform plane contact between a bulk superconductor and a thin normal conducting layer. Such model correctly catches the main physics of the proximity effect in spatially homogeneous structures in equilibrium. In particular, it accounts for the induced energy gap that depends on the contact resistance. However, it cannot describe spatially-dependent problems in 2D systems nor can it be easily extended to time-dependent or non-equilibrium phenomena.

In the present paper we develop an approach which

is suitable for spatially inhomogeneous and/or time-dependent problems associated with the induced superconductivity in 2D systems placed in a contact with a bulk superconductor. Our model is similar to that used in Refs.^{12,13} for impurities in a superconductor and is based on the so-called Fano–Anderson model which describes the decay of a localized state coupled to a continuum¹⁴. Our model can be applied to various 2D electronic systems, including simple 2D gas, graphene layer, etc. In this paper we consider two particular examples of junctions made of a 2D ballistic electron gas placed under the superconducting electrodes. The application of our model to the induced superconductivity in graphene will be considered elsewhere.

The paper is organized as follows. In the next Section we describe our proximity model in its general formulation suitable for various applications. The particular example of a junction between two proximity-induced superconductors is considered in Section III. We solve equations for the Green functions, find the energies of the bound states, and calculate the supercurrent through such junction. In Section IV we calculate the current-voltage curve for a contact between the semi-infinite normal and the proximity-induced superconducting regions. Our results are summarized in Section V.

II. MODEL

Consider a 2D electron layer placed under a bulk superconductor and coupled to it via tunnelling through a thin insulator coating, Fig. 1. The Hamiltonian of the system has the form

$$\hat{H} = \hat{H}_S + \hat{H}_{2D} + \hat{H}_T. \quad (1)$$

In the superconductor

$$\begin{aligned} \hat{H}_S = \int d^3r \left[\sum_{\alpha} \hat{\Psi}_{\alpha}^{\dagger}(\mathbf{r}) [\hat{\epsilon}_S - E_F] \hat{\Psi}_{\alpha}(\mathbf{r}) \right. \\ \left. + \Delta \hat{\Psi}_{\uparrow}^{\dagger}(\mathbf{r}) \hat{\Psi}_{\downarrow}^{\dagger}(\mathbf{r}) + \Delta^* \hat{\Psi}_{\downarrow}(\mathbf{r}) \hat{\Psi}_{\uparrow}(\mathbf{r}) \right], \quad (2) \end{aligned}$$

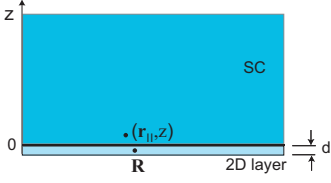


FIG. 1: Superconducting correlations in the 2D system under the superconductor are induced through a thin insulator.

where E_F is the chemical potential in the superconductor, $\hat{\epsilon}_s$ is the kinetic energy operator. For parabolic spectrum it is $\hat{\epsilon}_S = \frac{1}{2m} (-i\hbar\nabla - \frac{e}{c}\mathbf{A})^2$. The coordinate \mathbf{r} is a 3D vector which belongs to the superconductor. In the 2D layer

$$\hat{H}_{2D} = d \int d^2R \sum_{\alpha} \hat{a}_{\alpha}^{\dagger}(\mathbf{R}) [\hat{\epsilon}_{2D}(\mathbf{R}) - E_F] \hat{a}_{\alpha}(\mathbf{R}) , \quad (3)$$

where d is the layer thickness, $\hat{\epsilon}_{2D}(\mathbf{R})$ is the kinetic energy operator in the 2D layer. The coordinate \mathbf{R} is a 2D vector which belongs to the layer. The creation and annihilation operators in the layer $\hat{a}_{\alpha}^{\dagger}(\mathbf{R})$, $\hat{a}_{\alpha}(\mathbf{R})$ are normalized to the layer volume, such that the anti-commutator $[\hat{a}_{\alpha}(\mathbf{R}), \hat{a}_{\alpha}^{\dagger}(\mathbf{R}')]_{+} = d^{-1}\delta(\mathbf{R} - \mathbf{R}')$. The tunnel Hamiltonian has the form

$$\begin{aligned} \hat{H}_T = d \sum_{\alpha} \int \left[\hat{\Psi}_{\alpha}^{\dagger}(\mathbf{r}) T(\mathbf{r}, \mathbf{R}) \hat{a}_{\alpha}(\mathbf{R}) \right. \\ \left. + \hat{a}_{\alpha}^{\dagger}(\mathbf{R}) T^{\dagger}(\mathbf{R}, \mathbf{r}) \hat{\Psi}_{\alpha}(\mathbf{r}) \right] d^3r d^2R . \end{aligned} \quad (4)$$

Here the coordinate \mathbf{R} refers to the layer while coordinate \mathbf{r} refers to the superconductor; the matrix element $T(\mathbf{r}, \mathbf{R})$ describes tunnelling between the layer and the superconductor, $T^*(\mathbf{r}, \mathbf{R}) = T^{\dagger}(\mathbf{R}, \mathbf{r})$. This model is similar to the model used in Refs.^{12,13} for an impurity in superconductor.

The Matsubara Green functions are

$$\begin{aligned} \langle T_{\tau} \hat{a}_{\alpha}(\mathbf{R}_1) \hat{a}_{\beta}^{\dagger}(\mathbf{R}_2) \rangle &= \delta_{\alpha\beta} G_{2D}(\mathbf{R}_1, \mathbf{R}_2) , \\ \langle T_{\tau} \hat{\Psi}_{\alpha}(\mathbf{r}_1) \hat{a}_{\beta}^{\dagger}(\mathbf{R}_2) \rangle &= \delta_{\alpha\beta} G_T(\mathbf{r}_1, \mathbf{R}_2) , \\ \langle T_{\tau} \hat{\Psi}_{\alpha}(\mathbf{r}_1) \hat{\Psi}_{\beta}^{\dagger}(\mathbf{r}_2) \rangle &= \delta_{\alpha\beta} G_S(\mathbf{r}_1, \mathbf{r}_2) , \end{aligned}$$

and

$$\begin{aligned} \langle T_{\tau} \hat{a}_{\alpha}(\mathbf{R}_1) \hat{a}_{\beta}(\mathbf{R}_2) \rangle &= i\sigma_{\alpha\beta}^{(y)} F_{2D}(\mathbf{R}_1, \mathbf{R}_2) , \\ \langle T_{\tau} \hat{\Psi}_{\alpha}(\mathbf{r}_1) \hat{a}_{\beta}(\mathbf{R}_2) \rangle &= i\sigma_{\alpha\beta}^{(y)} F_T(\mathbf{r}_1, \mathbf{R}_2) , \\ \langle T_{\tau} \hat{\Psi}_{\alpha}(\mathbf{r}_1) \hat{\Psi}_{\beta}(\mathbf{r}_2) \rangle &= i\sigma_{\alpha\beta}^{(y)} F_S(\mathbf{r}_1, \mathbf{r}_2) , \end{aligned}$$

etc. We introduce the Nambu matrixes

$$\check{H}_S(\mathbf{r}) = \begin{pmatrix} \hat{\epsilon}_S - E_F & -\Delta \\ \Delta^* & \hat{\epsilon}_S - E_F \end{pmatrix} , \quad \check{G} = \begin{pmatrix} G & F \\ -F^{\dagger} & \bar{G} \end{pmatrix} ,$$

and denote

$$\check{G}_S^{-1}(\mathbf{r}) = \tilde{\tau}_3 \hbar \frac{\partial}{\partial \tau} + \check{H}_S(\mathbf{r})$$

the inverse operator in the superconductor. If needed, it can also include the impurity self energy. Neglecting the back-action of the thin 2D layer on the superconductor, we have for the superconducting Green function

$$\check{G}_S^{-1}(\mathbf{r}_1) \check{G}_S(\mathbf{r}_1, \mathbf{r}_2) = \check{1} \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\tau_1 - \tau_2) .$$

Equations for the mixed Green functions $\check{G}_T(\mathbf{r}_1, \mathbf{R}_2)$ can be written in the form

$$\check{G}_S^{-1}(\mathbf{r}_1) \check{G}_T(\mathbf{r}_1, \mathbf{R}_2) + d \int \check{T}(\mathbf{r}_1, \mathbf{R}') \check{G}_{2D}(\mathbf{R}', \mathbf{R}_2) d^2R' = 0$$

where

$$\check{T}(\mathbf{r}, \mathbf{R}) = \begin{pmatrix} T(\mathbf{r}, \mathbf{R}) & 0 \\ 0 & T^*(\mathbf{r}, \mathbf{R}) \end{pmatrix} .$$

This gives

$$\begin{aligned} \check{G}_T(\mathbf{r}_1, \mathbf{R}_2) \\ = -d \int \check{G}_s(\mathbf{r}_1, \mathbf{r}') \check{T}(\mathbf{r}', \mathbf{R}') \check{G}_{2D}(\mathbf{R}', \mathbf{R}_2) d^2R' d^3r' . \end{aligned} \quad (5)$$

Equations for the Green functions in the layer can be written as

$$\begin{aligned} \check{G}_{2D}^{-1}(\mathbf{R}_1) \check{G}_{2D}(\mathbf{R}_1, \mathbf{R}_2) + \int \check{T}^{\dagger}(\mathbf{R}_1, \mathbf{r}') \check{G}_T(\mathbf{r}', \mathbf{R}_2) d^3r' \\ = \check{1} d^{-1} \hbar \delta(\mathbf{R}_1 - \mathbf{R}_2) \delta(\tau_1 - \tau_2) \end{aligned}$$

where the inverse operator in the layer is introduced,

$$\check{G}_{2D}^{-1}(\mathbf{R}) = \hbar \tilde{\tau}_3 \frac{\partial}{\partial \tau} + [\hat{\epsilon}_{2D}(\mathbf{R}) - E_F] . \quad (6)$$

Here the kinetic energy operator is

$$\hat{\epsilon}_{2D} = \frac{1}{2m} \left(-i\hbar \frac{\partial}{\partial \mathbf{R}} - \tilde{\tau}_3 \frac{e}{c} \mathbf{A} \right)^2 + \epsilon_0 ,$$

while ϵ_0 is the shift of the bottom of the 2D conduction band measured from that in the superconductor.

Using Eq. (5) this equation becomes

$$\begin{aligned} \check{G}_{2D}^{-1}(\mathbf{R}_1) \check{G}_{2D}(\mathbf{R}_1, \mathbf{R}_2) - \int \check{\Sigma}_T(\mathbf{R}_1, \mathbf{R}') \\ \times \check{G}_{2D}(\mathbf{R}', \mathbf{R}_2) d^2R' = \check{1} \hbar d^{-1} \delta(\mathbf{R}_1 - \mathbf{R}_2) \delta(\tau_1 - \tau_2) \end{aligned} \quad (7)$$

where

$$\check{\Sigma}_T(\mathbf{R}_1, \mathbf{R}') = d \int \check{T}^{\dagger}(\mathbf{R}_1, \mathbf{r}') \check{G}_S(\mathbf{r}', \mathbf{r}'') \check{T}(\mathbf{r}'', \mathbf{R}') d^3r' d^3r'' .$$

We assume a site-to-site tunnelling which does not conserve momentum, $T(\mathbf{r}, \mathbf{R}) = t\delta(\mathbf{r}_{\parallel} - \mathbf{R})\delta(z)$, etc., where t is real and does not depend on \mathbf{R} . Here $\delta(z)$ selects an

average value of a function at a distance of the order of inter-atomic scale near the surface. As a result,

$$\check{\Sigma}_T(\mathbf{R}_i, \mathbf{R}_j) = dt^2 \begin{pmatrix} G_S(\mathbf{R}_i, \mathbf{R}_j; 0) & F_S(\mathbf{R}_i, \mathbf{R}_j; 0) \\ -F_S^\dagger(\mathbf{R}_i, \mathbf{R}_j; 0) & \bar{G}_S(\mathbf{R}_i, \mathbf{R}_j; 0) \end{pmatrix} \quad (8)$$

where $(\mathbf{R}_i; 0)$ means that $\mathbf{r}_{\parallel} = \mathbf{R}_i$ and $z_i = z_j = 0$.

The coordinate dependence of the Green functions has the form¹⁵

$$\check{G}_S(\mathbf{r}_i, \mathbf{r}_j) = \frac{me^{-|\mathbf{r}_i - \mathbf{r}_j|/\ell}}{2\pi\hbar^2|\mathbf{r}_i - \mathbf{r}_j|} [\tilde{1} \cos(p_F|\mathbf{r}_i - \mathbf{r}_j|/\hbar) + i \sin(p_F|\mathbf{r}_i - \mathbf{r}_j|/\hbar) \check{g}_S(\hat{\mathbf{n}}, \mathbf{r})] \quad (9)$$

Here ℓ is the mean free path in the superconductor, $\hat{\mathbf{n}} = (\mathbf{r}_i - \mathbf{r}_j)/|\mathbf{r}_i - \mathbf{r}_j|$ is a unit vector, $\mathbf{r} = (\mathbf{r}_i + \mathbf{r}_j)/2$, and \check{g}_S is the quasiclassical superconducting Green function integrated over the energy variable, Eq. (10),

$$\check{g}_S(\mathbf{p}_F, \mathbf{r}) = \int \frac{d\epsilon_s}{i\pi\hbar} \check{G}_S(\mathbf{p}, \mathbf{r}) \quad (10)$$

which depends only on the direction of the momentum $\mathbf{p}_F = p_F \mathbf{n}$.

The self-energy $\check{\Sigma}_T(\mathbf{R}_i, \mathbf{R}_l)$ in Eq. (8) oscillates as $\exp(ip_F|\mathbf{R}_j - \mathbf{R}_l|/\hbar)$ while the Green function contains $\exp[i\mathbf{p}(\mathbf{R}_l - \mathbf{R}_j)/\hbar] = \exp[i\mathbf{p}(\mathbf{R}_l - \mathbf{R}_j)/\hbar + i\mathbf{p}(\mathbf{R}_j - \mathbf{R}_l)/\hbar]$ where $|\mathbf{p}|$ is close to the Fermi momentum p_{2D} in the 2D layer. If $p_{2D} \neq p_F$, the integral in Eq. (7) converges at $R = |\mathbf{R}_j - \mathbf{R}_l| \sim a$ where a is the interatomic distance. Therefore, one can invoke the approximation

$$\Sigma_T(\mathbf{R}_j, \mathbf{R}_l) = \Sigma(\mathbf{R}_j) \delta(\mathbf{R}_j - \mathbf{R}_l) \quad .$$

The first term in Eq. (9) diverges for $|\mathbf{R} - \mathbf{R}'| \rightarrow 0$, but it does not depend on the superconducting state. It is real and has the form of an effective potential. We can include it into the shift of the bottom of the conduction band ϵ_0 or into the chemical potential assuming that μ already accounts for it. The remaining term for $|\mathbf{R} - \mathbf{R}'| \rightarrow 0$ gives

$$\check{\Sigma}(\mathbf{R}) = i\Gamma \langle \check{g}_S(\mathbf{R}; 0) \rangle \quad (11)$$

Here we introduce the tunnelling rate $\Gamma = \pi\nu_3 ds_0 t^2$ where $\nu_3 = mp_F/2\pi^2\hbar^3$ is the 3D density of states, $s_0 \sim a^2$ is of the order of the area of the unit cell in the layer; the overall order of magnitude is $\Gamma \sim t^2/E_F$. Angular brackets denote averaging over momentum directions. The quasiclassical Green function $\check{g}_S(\mathbf{R}; 0)$ taken at the superconductor/2D-layer interface is the only characteristic which is needed in our model to account of the properties of the bulk superconductor. Since the quasiclassical function varies over distances of the order of superconducting coherence length, the exact atomic-scale boundary conditions at the interface for the microscopic wave functions are not critical for our model.

We write the final equation in 2D layer for the real-time Green functions making the analytical continuation

of the Matsubara functions onto the real-frequency axis. We introduce the Keldysh matrixes

$$\mathcal{G} = \begin{pmatrix} \check{G}^R & \check{G}^K \\ 0 & \check{G}^A \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \check{\Sigma}^R & \check{\Sigma}^K \\ 0 & \check{\Sigma}^A \end{pmatrix} \quad .$$

The equations become (the index $2D$ is dropped)

$$(\epsilon_{2D} - E_F - \tilde{\tau}_3\epsilon) \mathcal{G}_{\epsilon, \epsilon'}(\mathbf{R}, \mathbf{R}') - [\mathcal{S}(\mathbf{R})\mathcal{G}(\mathbf{R}, \mathbf{R}')]_{\epsilon, \epsilon'} = i\hbar^2 d^{-1} \delta(\mathbf{R} - \mathbf{R}') 2\pi\delta(\epsilon - \epsilon') \quad (12)$$

where

$$[\mathcal{S}(\mathbf{R})\mathcal{G}(\mathbf{R}, \mathbf{R}')]_{\epsilon, \epsilon'} = \int \mathcal{S}_{\epsilon, \epsilon_1}(\mathbf{R}) \mathcal{G}_{\epsilon_1, \epsilon'}(\mathbf{R}, \mathbf{R}') \frac{d\epsilon_1}{2\pi\hbar} \quad .$$

III. S/2D/S STRUCTURE

A. Green functions

In this paper we restrict ourselves to stationary problems and put $\check{G}_{\epsilon, \epsilon'}^{R(A)} = \check{G}_{\epsilon}^{R(A)} 2\pi\hbar\delta(\epsilon - \epsilon')$. In the following Sections we apply our model to the S/2D/S structures which are made of a ballistic 2D electron gas placed in a contact with bulk superconducting leads (S). One of the examples is shown in Fig. 2. A generic structure of this type consists of a layer of 2D electron gas which is infinite in y direction and has boundaries at certain x where $\mathbf{R} = (x, y)$. Transforming to the plane waves along the y direction,

$$\check{G}_{\epsilon}^R(\mathbf{R}, \mathbf{R}') = \int e^{ip_y(y-y')/\hbar} \check{G}_{\epsilon, p_y}^R(x, x') \frac{dp_y}{2\pi\hbar} \quad ,$$

we observe that, as functions of x for $x \neq x'$, the retarded Green functions $G_{\epsilon, p_y}^R(x, x')$ and $F_{\epsilon, p_y}^{\dagger R}(x, x')$ satisfy the set of 2 linear homogeneous second-order differential equations which follow from Eq. (12),

$$\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \mu_x \right] u(x) - (\epsilon + \eta_1)u(x) + \eta_2 v(x) = 0, \quad (13)$$

$$\left[\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \mu_x \right] v(x) - (\epsilon + \eta_1)v(x) + \eta_2^\dagger u(x) = 0, \quad (14)$$

where $\mu = E_F - \epsilon_0$ is the chemical potential in the 2D gas measured from the bottom of the conduction band, $\mu_x = \mu - p_y^2/2m$, and

$$\eta_1 = i\Gamma \langle g_{\epsilon}^R \rangle, \quad \eta_2 = i\Gamma \langle f_{\epsilon}^R \rangle, \quad \eta_2^\dagger = i\Gamma \langle f_{\epsilon}^{\dagger R} \rangle \quad . \quad (15)$$

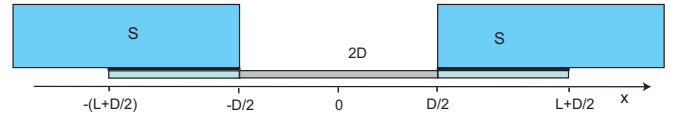


FIG. 2: Contact made of two superconducting electrodes placed on top of the 2D electron gas at a distance D from each other. Each electrode covers a length L of the 2D layer.

Eqs. (13), (14) have 4 independent solutions,

$$\check{\Psi}^{(i)}(x) = \begin{pmatrix} u^{(i)}(x) \\ v^{(i)}(x) \end{pmatrix}, \quad i = 1, 2, 3, 4.$$

Using the standard method in the theory of differential equations, one can write the Green functions as sums

$$\begin{pmatrix} G(x, x') \\ F^\dagger(x, x') \end{pmatrix} = C_1(x')\check{\Psi}^{(1)}(x) + C_2(x')\check{\Psi}^{(2)}(x), \quad x > x', \\ \begin{pmatrix} G(x, x') \\ F^\dagger(x, x') \end{pmatrix} = -C_3(x')\check{\Psi}^{(3)}(x) - C_4(x')\check{\Psi}^{(4)}(x), \quad x < x'.$$

The functions $\check{\Psi}^{(i)}(x)$ are chosen such that the retarded Green functions are regular in the upper half-plane, $\text{Im } \epsilon > 0$. The coefficients $C_k(x')$ do not depend on x but can be functions of x' . We find from Eq. (12)

$$\begin{aligned} G(x+0, x) - G(x-0, x) &= 0, \\ \frac{d}{dx}G(x+0, x) - \frac{d}{dx}G(x-0, x) &= -\frac{2m}{\hbar d}, \\ F^\dagger(x+0, x) - F^\dagger(x-0, x) &= 0, \\ \frac{d}{dx}F^\dagger(x+0, x) - \frac{d}{dx}F^\dagger(x-0, x) &= 0. \end{aligned}$$

Let us define the vectors

$$C_i = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix}, \quad R_k = -\frac{2m}{\hbar d} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the matrix U_k^i where

$$U_1^i = u^{(i)}, \quad U_2^i = du^{(i)}/dx, \quad U_3^i = v^{(i)}, \quad U_4^i = dv^{(i)}/dx.$$

The upper index numerates columns, while the lower numerates rows. Equations take the form $\sum_i U_k^i C_i = R_k$. The solution is

$$C_i(x) = W^{-1} \sum_k A_k^i(x) R_k. \quad (16)$$

Here $W = \det[U_k^i]$ is the Wronskian and $A_k^i(x) = (-1)^{i+k} W_k^i(x)$ where $W_k^i(x)$ is a minor of $\det[U_k^i]$. The Wronskian is independent of coordinates.

B. Reflection coefficients

Consider structure shown in Fig. 2. The right superconductor has the phase $\phi/2$ while the left one has $-\phi/2$. The four independent functions needed for the Green functions can be constructed as follows. The first two wave functions inside the 2D layer $-D/2 < x < D/2$ contain waves incident from the left and reflected from the right boundary,

$$\begin{aligned} \check{\Psi}^{(1)}(x) &= e^{ik_+(x-\frac{D}{2})}\check{\Psi}_p + r_A^R e^{ik_-(x-\frac{D}{2})}\check{\Psi}_h \\ &\quad + r_N^R e^{-ik_+(x-\frac{D}{2})}\check{\Psi}_p, \end{aligned} \quad (17)$$

$$\begin{aligned} \check{\Psi}^{(2)}(x) &= e^{-ik_-(x-\frac{D}{2})}\check{\Psi}_h + \bar{r}_A^R e^{-ik_+(x-\frac{D}{2})}\check{\Psi}_p \\ &\quad + \bar{r}_N^R e^{ik_-(x-\frac{D}{2})}\check{\Psi}_h. \end{aligned} \quad (18)$$

Here $k_\pm = k_x \pm \epsilon/\hbar v_x$, while k_x and v_x are the x components of the Fermi wave vector and Fermi velocity $\hbar k_x = mv_x$ in the 2D gas, $\hbar^2 k_x^2/2m = \mu_x$; the particle and hole Nambu vectors are

$$\check{\Psi}_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \check{\Psi}_h = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The set of two other functions is obtained using reflection from the left boundary,

$$\begin{aligned} \check{\Psi}^{(3)}(x) &= e^{-ik_+(x+\frac{D}{2})}\check{\Psi}_p + r_A^L e^{-ik_-(x+\frac{D}{2})}\check{\Psi}_h \\ &\quad + r_N^L e^{ik_+(x+\frac{D}{2})}\check{\Psi}_p, \end{aligned} \quad (19)$$

$$\begin{aligned} \check{\Psi}^{(4)}(x) &= e^{ik_-(x+\frac{D}{2})}\check{\Psi}_h + \bar{r}_A^L e^{ik_+(x+\frac{D}{2})}\check{\Psi}_p \\ &\quad + \bar{r}_N^L e^{-ik_-(x+\frac{D}{2})}\check{\Psi}_h. \end{aligned} \quad (20)$$

In the right induced-superconductivity region $D/2 < x < L + D/2$, the four functions satisfying Eqs. (13) - (14) are linear combinations of

$$\begin{aligned} e^{ik_+^s(x-L-\frac{D}{2})} \begin{pmatrix} ue^{i\frac{\phi}{4}} \\ ve^{-i\frac{\phi}{4}} \end{pmatrix}, \quad e^{-ik_+^s(x-L-\frac{D}{2})} \begin{pmatrix} ue^{i\frac{\phi}{4}} \\ ve^{-i\frac{\phi}{4}} \end{pmatrix}, \\ e^{ik_-^s(x-L-\frac{D}{2})} \begin{pmatrix} ve^{i\frac{\phi}{4}} \\ ue^{-i\frac{\phi}{4}} \end{pmatrix}, \quad e^{-ik_-^s(x-L-\frac{D}{2})} \begin{pmatrix} ve^{i\frac{\phi}{4}} \\ ue^{-i\frac{\phi}{4}} \end{pmatrix}. \end{aligned}$$

The functions in the left superconducting region are obtained by replacing ϕ with $-\phi$ and $(L + D/2)$ with $-(L + D/2)$. Note that the momentum in the induced-superconductivity region,

$$k_\pm^s = k_x \pm \frac{1}{\hbar v_x} \sqrt{(\epsilon + \eta_1)^2 - \eta_2 \eta_2^\dagger}, \quad (21)$$

has the same Fermi-momentum projection on the x coordinate as in the normal region. Thus the normal reflection at $x = \pm D/2$ does not occur. The coherence factors are

$$u(\epsilon) = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi}{\epsilon + \eta_1}}, \quad v(\epsilon) = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi}{\epsilon + \eta_1}}. \quad (22)$$

Here

$$\xi^2 = (\epsilon + \eta_1)^2 - \eta_2 \eta_2^\dagger = \tilde{\epsilon}^2 - \Gamma^2. \quad (23)$$

since $\eta_1^2 - \eta_2 \eta_2^\dagger = -\Gamma^2$. Moreover

$$g^R = \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}, \quad f^R = \frac{\Delta}{\sqrt{\epsilon^2 - \Delta^2}}, \quad \tilde{\epsilon}^2 = \epsilon^2 + \frac{2i\Gamma\epsilon^2}{\sqrt{\epsilon^2 - |\Delta|^2}}.$$

The square roots are defined as analytic functions in the plane of complex ϵ with cuts from $|\Delta|$ to ∞ and from $-\infty$ to $-|\Delta|$ taken at the upper bank of the cut for $\epsilon > |\Delta|$. The usual BCS formulas are recovered if one uses $\epsilon \rightarrow \epsilon + \eta_1$, $\Delta \rightarrow \eta_2$, $\Delta^* \rightarrow \eta_2^\dagger$ such that the BCS gap $|\Delta|^2$ is replaced with $\Gamma^2 |\Delta|^2 / (|\Delta|^2 - \epsilon^2)$.

The boundary conditions require that $u^{(i)} = v^{(i)} = 0$ at $x = \pm(L + D/2)$ together with continuity of $u^{(i)}, v^{(i)}, du^{(i)}/dx, dv^{(i)}/dx$ at $x = \pm D/2$. This yields

$$r_N^R = r_N e^{i\delta}, \bar{r}_N^R = r_N e^{-i\delta}, \quad (24)$$

$$r_A^R = r_A e^{-i\phi/2}, \bar{r}_A^R = r_A e^{i\phi/2}, \quad (25)$$

$$r_N^L = r_N e^{i\delta}, \bar{r}_N^L = r_N e^{-i\delta}, \quad (26)$$

$$r_A^L = r_A e^{i\phi/2}, \bar{r}_A^L = r_A e^{-i\phi/2}, \quad (27)$$

where

$$r_N = -\frac{(u^2 - v^2)e^{i(\delta_+ - \delta_-)}}{u^2 - v^2 e^{2i(\delta_+ - \delta_-)}}, \quad r_A = \frac{uv[1 - e^{2i(\delta_+ - \delta_-)}]}{u^2 - v^2 e^{2i(\delta_+ - \delta_-)}}.$$

Here we denote $\delta_{\pm} = k_{\pm}^s L$ and $\delta = \delta_+ + \delta_- = 2k_x L$, while

$$\delta_+ - \delta_- = (2L/\hbar v_x) \sqrt{\tilde{\epsilon}^2 - \Gamma^2}. \quad (28)$$

One can check that the Green functions defined according to Eq. (16) with the basis functions Eqs. (17)–(20) are regular in the upper half-plane of complex ϵ . For $\epsilon < \Delta$ the coefficients satisfy

$$|r_N|^2 + |r_A|^2 = 1, \quad r_N^* r_A + r_A^* r_N = 0 \quad (29)$$

because there is no quasiparticle flux into the superconductor.

As in Ref.¹⁶ one can define an equivalent barrier height Z associated with the end of the conduction layer,

$$Z/\sqrt{1 + Z^2} = e^{i(\delta_+ - \delta_-)}, \quad (30)$$

such that the coefficients have their standard form^{17,18}

$$r_A = \frac{uv}{u^2 + (u^2 - v^2)Z^2}, \quad r_N = -\frac{(u^2 - v^2)Z\sqrt{1 + Z^2}}{u^2 + (u^2 - v^2)Z^2}. \quad (31)$$

The limit $L = 0$ corresponds to strong normal reflection, $Z \rightarrow \infty$ and $|r_N| = 1$, while $r_A = 0$. With this definition of effective barriers, the problems of the bound-state spectrum and of the supercurrent in many respects reduce to the corresponding problems in double-barrier superconductor/normal/superconductor (SINIS) structures.

Let us denote $\epsilon_g < \Delta$ the energy at which the square root in Eq. (21) vanishes, i.e., at which ξ defined by Eq. (23) turns to zero,

$$\epsilon_g^2 \left(1 + 2\Gamma/\sqrt{\Delta^2 - \epsilon_g^2}\right) - \Gamma^2 = 0. \quad (32)$$

The energy ϵ_g is the induced gap in the 2D layer. For $\Gamma \ll \Delta$ we have $\epsilon_g = \Gamma$; if $\Gamma \gg \Delta$ one has $\epsilon_g = \Delta(1 - 2\Delta^2/\Gamma^2)$. The induced gap as a function of Γ is shown in Fig. 3. For a 2D gas in a direct contact with a superconductor, the induced gap was obtained in Ref.¹⁰.

If $\epsilon^2 < \epsilon_g^2$, both the square root in Eq. (21) and ξ are imaginary. The value Z is real and satisfies

$$Z/\sqrt{1 + Z^2} = \exp\left[-2(L/\hbar v_x)\sqrt{\Gamma^2 - \tilde{\epsilon}^2}\right]. \quad (33)$$

For large L we have $Z = 0$; the particles do not feel the dead end of the superconducting region, such that the normal reflection is absent, while $|r_A| = 1$. This limit is realized when $L \gg \xi_{2D}$ where

$$\xi_{2D} = \hbar v_{2D}/\epsilon_g$$

is the coherence length of the induced superconductivity, v_{2D} is the Fermi velocity in the 2D system $v_{2D}^2/2m = \mu$. The limit of long leads is rather hard to realize because ξ_{2D} is considerably longer than ξ_0 in a bulk superconductor. On the contrary, for short leads, $L = 0$, there exists only normal reflection, $Z \rightarrow \infty$, and $|r_N| = 1$.

If $\epsilon_g < \epsilon < |\Delta|$ both the phases δ_{\pm} and the coherence factors u, v are real, while Z is complex. The Andreev reflection vanishes while $|r_N| = 1$ if $\delta_+ - \delta_- = \pi n$.

For $\epsilon > |\Delta|$ the coefficients still have the form of Eq. (31) with a complex effective barrier strength Z . For large $\epsilon \gg |\Delta|$ one has $\tilde{\epsilon} = \epsilon + i\Gamma$. The effective barrier height saturates at $Z \neq 0$, while $v \rightarrow 0$ and thus $r_A \rightarrow 0$. At the same time

$$|r_N|^2 = \exp[-4L\Gamma/\hbar v_x].$$

Equation (29) no longer holds because the quasiparticles escape into the superconductor.

C. Bound states. Supercurrent

Using the functions Eqs. (17) – (20) and Eqs. (24)–(27), we find the Wronskian

$$W = 16k_+ k_- e^{2i\gamma} [\sin^2(\beta + \gamma) - |r_A|^2 \cos^2(\phi/2) - |r_N|^2 \sin^2(\alpha + \delta)]$$

where $\alpha = k_x D$, $\beta = \epsilon D/\hbar v_x$, while γ is the phase of the reflection coefficient r_N , i.e., $e^{2i\gamma} = r_N/r_N^*$. The spectrum is determined by $W = 0$, which gives

$$\sin^2(\beta + \gamma) = |r_A|^2 \cos^2(\phi/2) + |r_N|^2 \sin^2 \alpha' \quad (34)$$

where $\alpha' = \alpha + \delta$. This agrees with the previous calculations of Ref.¹⁸ for ballistic SINIS contacts. Here we concentrate on short contacts such that $\beta \ll 1$. Neglecting β and keeping L finite implies that one has to assume $L \gg D$.

If the energy is above the induced gap, $\epsilon_g < \epsilon < \Delta$, the combination $\delta_+ - \delta_-$, as well as u and v are real.

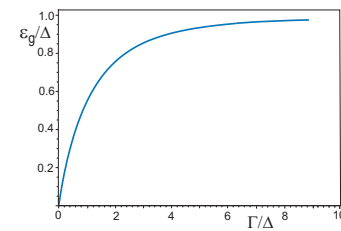


FIG. 3: Induced gap as a function of the tunneling rate.

For energies below the induced gap, $\epsilon < \epsilon_g$, the barrier strength Z , Eq. (30), is real, while $u = v^*$. Therefore, Eq. (34) yields for the bound state energy

$$\frac{(\epsilon + \eta_1)^2}{\eta_2 \eta_2^\dagger} = 1 - \frac{\sin^2(\phi/2)}{1 + A} \quad (35)$$

where

$$A = -\frac{\sin^2 \alpha'}{\sin^2[2L\sqrt{\tilde{\epsilon}^2 - \Gamma^2}/\hbar v_x]}, \quad \epsilon_g < \epsilon < \Delta \quad (36)$$

$$A = \frac{\sin^2 \alpha'}{\sinh^2[2L\sqrt{\Gamma^2 - \tilde{\epsilon}^2}/\hbar v_x]}, \quad \epsilon < \epsilon_g \quad (37)$$

Equation (36) goes into Eq. (37) if one continues analytically the phase difference $\delta_+ - \delta_-$ as a function of ϵ around the induced gap ϵ_g .

For energies below Δ quasiparticles cannot escape into the superconductors and form the bound states as a result of two processes: the Andreev reflection at the boundary between the normal region and the region with induced-superconductivity plus the normal reflection at the ends of the 2D layer. The spectrum is determined by Eq. (35); it consists of a series of levels for each p_y and ϕ spaced with a distance $\sim \hbar v_x/L$ and filling the interval $0 < \epsilon < \Delta$. The lowest energy level lies below the induced gap if $(2L\Gamma/\hbar v_x)\sin(\phi/2) > \sin \alpha'$. For short leads, $2L\Gamma/\hbar v_x \ll \sin \alpha'$, the energies are close to the levels of geometrical quantization in a potential well of length $d + 2L$, i.e., they satisfy $\delta_+ - \delta_- \pm \alpha' = \pi n$. If $L \ll \hbar v_x/\Delta$, there is only one bound state with energy $\epsilon < \Delta$.

The supercurrent is¹⁹

$$I = -\frac{\pi}{eR_0} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta \sum_n \frac{\partial \epsilon_n}{\partial \phi} \tanh \frac{\epsilon_n}{2T}. \quad (38)$$

Here

$$R_0^{-1} = 2e^2 w \nu_2 v_{2D} / \pi = G_0 w p_{2D} / \pi \hbar$$

is the Sharvin conductance of the contact, $G_0 = e^2/\pi \hbar$ is the conductance quantum, $p_{2D} = m v_{2D}$ is the Fermi momentum in the layer, and $\nu_2 = m/2\pi \hbar^2$ is the normal density of states in the 2D gas. The conductance is proportional to the number of modes in the 2D channel of width w in the y direction. The sum collects all bound states with energies $\epsilon < \Delta$ for given $p_y = p_{2D} \sin \theta$. The continuum states with $\epsilon > \Delta$ give the contribution proportional to the length D and can thus be neglected.

1. Short leads

Consider the case of short leads, $L\Gamma/\hbar v_x \ll 1$ for $\Gamma \ll \Delta$. This is the most practical situation in the experimental devices. If $\hbar v_x/\Delta \ll L \ll \hbar v_x/\Gamma$, there is a series of bound states with energies $0 < \epsilon_n < \Delta$. As discussed

above, they are close to the levels of dimensional quantization. Putting $x = (2L\Gamma/\hbar v_x)\sqrt{\tilde{\epsilon}^2/\Gamma^2 - 1}$ we observe that $x = x_n + \delta x_n$ where

$$\sin^2 x_n = \sin^2 \alpha', \quad (39)$$

and

$$\delta x_n = -\left(\frac{2L\Gamma}{\hbar v_x}\right)^2 \sin^2(\phi/2) \frac{\tan x_n}{2x_n^2} \quad (40)$$

which follows from Eqs. (35), (36). Equation (39) gives

$$x_n = \begin{cases} \pm x_0 + \pi n, & n = 1, 2, \dots, \\ x_0, & n = 0. \end{cases} \quad (41)$$

Here $x_0 = \arcsin |\sin \alpha'|$ is the smallest root, $0 < x_0 < \pi/2$, of Eq. (39). This solution holds as long as $\delta x_n \ll x_n$, i.e., for all states with $n \neq 0$. It also holds for the lowest energy state, as long as the level is not very close to the resonance, $|\sin \alpha'| \gg (2L\Gamma/\hbar v_x)$.

Close to the resonance, $|\sin \alpha'| \ll 1$, the lowest level can also be obtained by expanding the sine functions in Eqs. (36), (37). Putting $\tilde{\epsilon} \approx \epsilon$ we find

$$\epsilon^2 = \frac{\Delta^2 + \Gamma^2(1 + a^2)}{2} - \sqrt{\frac{[\Delta^2 - \Gamma^2(1 + a^2)]^2}{4} + \Delta^2 \Gamma^2 \sin^2(\phi/2)}, \quad (42)$$

where $a^2 = \hbar^2 v_x^2 \sin^2 \alpha' / (2L\Gamma)^2$. Close to the resonance, $a\Gamma \ll \Delta$,

$$\epsilon = \Gamma \sqrt{a^2 + \cos^2(\phi/2)}. \quad (43)$$

This goes over into Eqs. (41), (40) when $a \gg 1$. The typical spectrum is shown in Fig. 4(a).

If the leads are very short, $L \ll \hbar v_x/\Delta$ then there exists only one level. Except for a very narrow vicinity of resonance, one has $a^2 \Gamma^2 \gg \Delta^2$, thus the level is close to the superconducting gap Δ ,

$$\epsilon^2 = \Delta^2 - (\Delta^2/a^2) \sin^2(\phi/2). \quad (44)$$

To calculate the supercurrent Eq. (38) we integrate the contribution from each level over the incident angle θ defined according to $v_x = v_{2D} \cos \theta$. Since the resonance form of the level, Eq. (43), holds only in a very narrow region of angles, its contribution to the current is small. The current is thus mostly determined by the off-resonance levels, Eqs. (41), (40), or (44) with $\epsilon_n \gg \Gamma$. For $\hbar v_x/\Delta \ll L \ll \hbar v_x/\Gamma$, there is a series of levels defined by Eqs. (41), (40). The current is

$$I = \frac{\pi}{eR_0} \left(\frac{L\Gamma^2}{\hbar v_{2D}} \right) \sin \phi \int_0^{\pi/2} \tan x_0 \times \left[F(x_0) + \sum_{n=1}^{\infty} [F(\pi n + x_0) - F(\pi n - x_0)] \right] d\theta \quad (45)$$

where

$$F(x) = \frac{1}{x^2} \tanh \frac{\hbar v_x x}{4LT}.$$

Due to a large argument in $\sin(\alpha') = \sin[k_{2D}(d + 2L)\cos\theta]$, the level x_0 oscillates rapidly assuming values within the interval $0 < x_0 < \pi/2$ many times as the incident angle θ varies from 0 to $\pi/2$. Let us consider a function $f(x_0, \theta)$ of the rapidly oscillating variable $x_0(\theta)$ and a slow variable θ and define the average function $\langle f(\theta) \rangle \delta\theta = \int_{\theta}^{\theta+\delta\theta} f(x_0, \theta) d\theta$ where the integral is taken over the full variation range of x_0 putting $dx_0 = (d + 2L)k_{2D} \sin\theta d\theta$. Since the range $0 < x_0 < \pi/2$ corresponds to a small variation $\delta\theta \ll 1$, one can keep the slow variable θ constant during integration over dx_0 ,

$$\langle f(\theta) \rangle = \frac{2}{\pi} \int_0^{\pi/2} f(x_0, \theta) dx_0.$$

The integral of the rapid function $f(x_0, \theta)$ can now be replaced with the integral of the average function $\int_0^{\pi/2} f(x_0, \theta) d\theta = \int_0^{\pi/2} \langle f(\theta) \rangle d\theta$.

For low temperatures, $T \ll \hbar v_{2D}/L$ one can simplify the expression (45) for the current. We note that for $n \neq 0$ the integral in $\langle \tan x_0 F(\pi n \pm x_0) \rangle$ diverges logarithmically at $x_0 \rightarrow \pi/2$; it should be cut off when $\delta x_n \sim 1$, i.e., at $\pi/2 - x_0 \sim (L\Gamma \sin(\phi/2)/\hbar v_x)^2$. Therefore, within the logarithmic accuracy,

$$\begin{aligned} \langle \tan x_0 F(\pi n \pm x_0) \rangle &= \left(\frac{4}{\pi}\right)^3 \ln\left(\frac{\hbar v_{2D}}{L\Gamma}\right) \\ &\times \frac{1}{(2n \pm 1)^2} \tanh \frac{\pi \hbar v_x (2n \pm 1)}{8LT}. \end{aligned}$$

For $n = 0$ the integral diverges logarithmically at $x_0 \rightarrow \pi/2$. At the lower limit, $x_0 \rightarrow 0$, the function $F(x) \tan x$ under the integral should be replaced with that containing the resonance level taken from Eq. (43). Therefore, we find

$$\langle \tan x_0 F(x_0) \rangle = \left(\frac{4}{\pi}\right)^3 \ln\left(\frac{\hbar v_{2D}}{L\Gamma}\right) \tanh \frac{\pi \hbar v_x}{8LT} + \frac{2}{\pi} B(\theta)$$

where

$$\begin{aligned} B(\theta) &= \int_{x'_0}^{\frac{\pi}{2}} \left[\frac{1}{x_0^2} \tanh \frac{\hbar v_x x_0}{4LT} - \frac{4}{\pi^2} \tanh \frac{\pi \hbar v_x}{8LT} \right] \tan x_0 dx_0 \\ &\quad + \int_0^{x'_0} \frac{dx_0}{\tilde{x}_0} \tanh \frac{\hbar v_x \tilde{x}_0}{4LT}. \end{aligned}$$

Here $\tilde{x}_0 = \sqrt{x_0^2 + (2L\Gamma/\hbar v_x)^2 \cos^2(\phi/2)}$ and $L\Gamma/\hbar v_{2D} \ll x'_0 \ll 1$. For low temperatures,

$$B = \begin{cases} \ln\left(\frac{\hbar v_{2D}}{L\Gamma|\cos(\phi/2)|}\right), & T \ll \Gamma|\cos(\phi/2)| \\ \ln\left(\frac{\hbar v_{2D}}{LT}\right), & \Gamma|\cos(\phi/2)| \ll T \ll \frac{\hbar v_{2D}}{L} \end{cases}$$

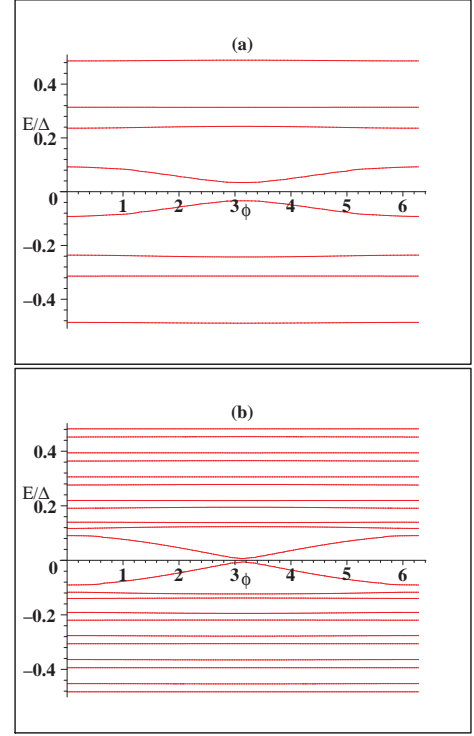


FIG. 4: Spectrum of bound states in a Josephson junction as a function of superconducting phase difference. Here we put (a) $\Gamma/\Delta = 0.1$, $2L\Delta/\hbar v_F = 10$, $\hbar k_F v_F/\Delta = 10$, $D\Delta/\hbar v_F = 1$, $\theta = 0$; (b) $\Gamma/\Delta = 0.1$, $2L\Delta/\hbar v_F = 30$, $\hbar k_F v_F/\Delta = 10$, $D\Delta/\hbar v_F = 1$, $\theta = 0$

It is independent of θ within the logarithmic accuracy. Performing the summation we find for $T \ll v_{2D}/L$

$$I = \frac{\pi}{eR_0} \left(\frac{L\Gamma^2}{\hbar v_{2D}} \right) B \sin \phi. \quad (46)$$

The current-phase relation is sinusoidal; this corresponds to the limit of low transparency junctions $Z \gg 1$ realized in short-lead contacts with $L\Gamma/\hbar v_{2D} \ll 1$. The supercurrent increases with lowering the temperature and saturates for $T \lesssim \Gamma$. For temperatures higher than $\hbar v_{2D}/L$, the logarithm decreases to a value of order unity, and our approximation breaks down. Note that the current-phase relation, Eq. (46), is similar to that obtained for a double barrier SINIS structure with resonant transmission (see, e.g., Ref. 20).

If the leads are very short, $L \ll \hbar v_{2D}/\Delta$, there exists only one level, Eq. (42). The estimates show that, within the logarithmic approximation, the current is dominated by the spectrum close to the resonance, Eq. (43), and still has the form of Eq. (46).

2. Long leads

Here we again restrict ourselves to the limit $\Gamma \ll \Delta$. For the contact with long leads, $L\Gamma/\hbar v_x \gg 1$ the spec-

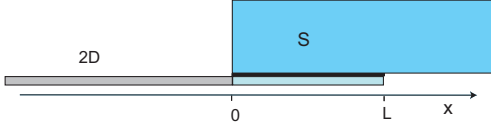


FIG. 5: Junction made of the superconducting electrode placed on top of the 2D electron gas.

trum consists of one level with $\epsilon_0 < \Gamma$ satisfying Eqs. (35), (37) and a series of levels with $\epsilon_n > \Gamma$ satisfying Eqs. (35), (36). Typical spectrum of bound states for a contact with long leads is plotted in Fig. 4(b).

For $\epsilon < \Gamma$ we find from Eqs. (35), (37)

$$x = \frac{(2L\Gamma/\hbar v_x) \sin(\phi/2) \sinh x}{\sqrt{\sin^2 \alpha' + \sinh^2 x}} \quad (47)$$

where $x = (2L\Gamma/\hbar v_x) \sqrt{1 - \epsilon^2/\Gamma^2}$. If $L\Gamma/\hbar v_x \gg 1$ we have $\sinh x \gg 1$ thus $x = (2L\Gamma/\hbar v_x) \sin(\phi/2)$, i.e., this energy state is as in a ballistic junction with a gap Γ ,

$$\epsilon^2 = \Gamma^2 \cos^2(\phi/2).$$

This level gives the usual expression for the supercurrent as in a ballistic contact²¹ with an induced gap Γ^{10} ,

$$I = \frac{\pi\Gamma \sin(\phi/2)}{eR_0} \tanh \frac{\Gamma \cos(\phi/2)}{2T}. \quad (48)$$

One also needs to consider the contribution of levels with energies above the induced gap. For $L\Gamma/\hbar v_x \gg 1$ Eqs. (35) and (36) yield $x' = \pi n + \delta x_n$ where $x' = (2L\Gamma/\hbar v_x) \sqrt{\epsilon^2/\Gamma^2 - 1}$ and

$$\pi n + \delta x_n = b |\sin(\delta x_n)|. \quad (49)$$

Here we denote $b^2 = (2L\Gamma/\hbar v_x)^2 \sin^2(\phi/2)/\sin^2 \alpha'$. Expanding in $b^{-1} \ll 1$ we find $\delta x_n = \pm \pi n/b - \pi n/b^2$. For $\epsilon \gg \Gamma$ we have

$$\sum_n \frac{\partial \epsilon_n}{\partial \phi} = -\frac{\hbar v_x}{2L} \sum_n \left[\pi n \frac{\partial}{\partial \phi} \frac{1}{b^2} \right].$$

The terms in δx proportional to b^{-1} disappear. The number of terms in the sum is $N \sim b$, being determined by the condition $\delta x \lesssim 1$. Therefore, the sum over the states with $\epsilon > \Gamma$ is of the order of $(\hbar v_x/2L) \ll \Gamma$, and can thus be neglected. As a result, Eq. (48) is the full expression for supercurrent through a contact with long leads. In this sense, the contact with long leads behaves as a contact without normal reflection, $|r_N| = 0$, when the bound states would only exist for $\epsilon < \epsilon_g$. This limit was considered in Ref.¹⁰.

IV. THE I-V CURVE IN A 2D/S JUNCTION

The results obtained in the previous section can be used to describe the transport in a 2D/S junction shown

in Fig. 5. The junction consists of a semi-infinite normal 2D layer a part of which (with a length L) is covered by a bulk superconducting lead. The current through such contact can be written in terms of reflection coefficients obtained in Sec. IIIB. Using the results of Ref.¹⁷, we have

$$I = \frac{1}{eR_0} \int_{-\infty}^{\infty} d\epsilon \int_{v_x > 0} \frac{d\theta}{2} \frac{v_x}{v_{2D}} [1 - |r_N|^2 + |r_A|^2] \times [f_0(\epsilon - eV) - f_0(\epsilon)]$$

where $f_0(\epsilon) = [e^{\epsilon/T} + 1]^{-1}$ is the Fermi function. For $T \ll \Gamma$ the differential conductance is

$$\frac{dI}{dV} = \frac{1}{R_0} \int_0^{\pi/2} \cos \theta [1 - |r_N|^2 + |r_A|^2] d\theta, \quad (50)$$

where $\epsilon = eV$.

A. Long leads. Andreev reflection

Consider the case $L \rightarrow \infty$ such that the length is longer than the electronic mean free path, $L \gg \ell$. In this limit one can neglect the effect of the dead end and assume that $Z = 0$.

For $Z = 0$ we have zero normal reflection amplitude $r_N = 0$ and the Andreev reflection coefficient $|r_A|^2 = |v|^2/|u|^2$ which is independent of p_y . For $\epsilon > \Delta$ the Andreev reflection coefficients becomes

$$|r_A|^2 = \left| \frac{\epsilon \left(1 + \frac{i\Gamma}{\sqrt{\epsilon^2 - \Delta^2}} \right) - \sqrt{\epsilon^2 \left(1 + \frac{2i\Gamma}{\sqrt{\epsilon^2 - \Delta^2}} \right) - \Gamma^2}}{\epsilon \left(1 + \frac{i\Gamma}{\sqrt{\epsilon^2 - \Delta^2}} \right) + \sqrt{\epsilon^2 \left(1 + \frac{2i\Gamma}{\sqrt{\epsilon^2 - \Delta^2}} \right) - \Gamma^2}} \right|$$

For $\epsilon < \Delta$ we have

$$|r_A|^2 = \left| \frac{\epsilon \left(1 + \frac{\Gamma}{\sqrt{\Delta^2 - \epsilon^2}} \right) - \sqrt{\epsilon^2 \left(1 + \frac{2\Gamma}{\sqrt{\Delta^2 - \epsilon^2}} \right) - \Gamma^2}}{\epsilon \left(1 + \frac{\Gamma}{\sqrt{\Delta^2 - \epsilon^2}} \right) + \sqrt{\epsilon^2 \left(1 + \frac{2\Gamma}{\sqrt{\Delta^2 - \epsilon^2}} \right) - \Gamma^2}} \right|$$

If $\epsilon^2 < \epsilon_g^2$, one has $|u| = |v|$ and $|r_A|^2 = 1$. If $\epsilon \rightarrow \Delta - 0$ we also have $|r_A|^2 \rightarrow 1$. If $\Gamma \ll \Delta$ then for $\epsilon_g \ll \epsilon \ll \Delta$ we have $|r_A|^2 = \Gamma^2/4\epsilon^2$. Since the Andreev reflection does not depend on θ , Eq. (50) yields for $T \ll \Gamma$

$$R_0 \frac{dI}{dV} = 1 + |r_A|^2$$

where $\epsilon = eV$. The Andreev reflection coefficient for a long contact is shown in Fig. 6. The limit of a long contact with zero normal reflection was considered in Ref.¹¹ using a direct-contact model.

B. Finite-length leads. Oscillations of the resistance

Here we discuss contacts with finite-length leads in the weak coupling limit, $\Gamma \ll |\Delta|$. For $\epsilon < |\Delta|$ when quasiparticles cannot escape into the leads, Eq. (29) gives $1 - |r_N|^2 + |r_A|^2 = 2|r_A|^2$. Equation (50) yields

$$R_0 \frac{dI}{dV} = 2 \int_0^{\pi/2} \cos \theta |r_A(\epsilon)|^2 d\theta.$$

In the low-voltage region, $\epsilon < \epsilon_g$, we have $u^* = v$ while Z is real. Therefore,

$$|r_A|^2 = \frac{4u^2v^2 \sinh^2[2L\sqrt{\Gamma^2 - \tilde{\epsilon}^2}/\hbar v_x]}{4u^2v^2 \cosh^2[2L\sqrt{\Gamma^2 - \tilde{\epsilon}^2}/\hbar v_x] - 1}.$$

In the weak coupling case $\epsilon_g = \Gamma$ and

$$\eta_1 = \epsilon\Gamma/\Delta, \quad \eta_2 = \Gamma, \quad 4u^2v^2 = \Gamma^2/\epsilon^2, \quad \tilde{\epsilon} = \epsilon.$$

The differential conductance becomes

$$R_0 \frac{dI}{dV} = 2 \times \begin{cases} \frac{4L^2}{\xi_{2D}^2} \ln \frac{\hbar v_{2D}}{L\Gamma}, & L\Gamma/\hbar v_{2D} \ll 1 \\ 1, & L\Gamma/\hbar v_{2D} \gg 1 \end{cases}$$

It does not depend on voltage. For short leads $L\Gamma/v_{2D} \ll 1$, the conductance is determined by the incident angles θ close to $\pi/2$ where the Andreev reflection is of the order of unity. For long leads $L\Gamma/\hbar v_{2D} \gg 1$ the Andreev reflection is complete as in the previous Section IV A.

For energies $\epsilon_g \ll \epsilon < |\Delta|$, the Andreev coefficient is

$$|r_A|^2 = \frac{2u^2v^2[1 - \cos[2(\delta_+ - \delta_-)]]}{u^4 + v^4 - 2u^2v^2 \cos[2(\delta_+ - \delta_-)]}.$$

If the energy is not very close to the superconducting gap, $|\epsilon - \Delta| \gg \Gamma^2/\Delta$, we have $v^2 \ll u^2$, while $u^2 = 1$. Therefore

$$|r_A|^2 = 2v^2[1 - \cos[2(\delta_+ - \delta_-)]]$$

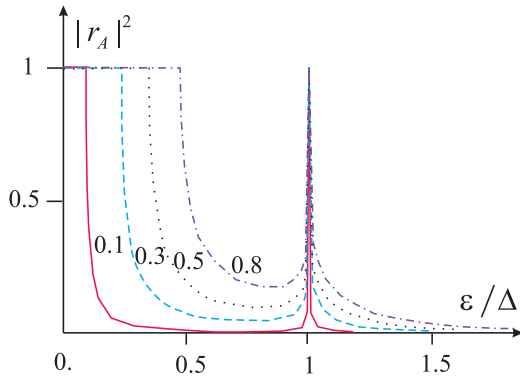


FIG. 6: Andreev reflection coefficient as a function of energy for zero normal reflection, $Z = 0$. Here $x = \epsilon/|\Delta|$.

and $v^2(\epsilon) = \Gamma^2/4\epsilon^2$. Therefore, for $\epsilon_g \ll eV < \Delta$,

$$R_0 \frac{dI}{dV} = \frac{4L^2}{\xi_{2D}^2} \times \begin{cases} 2 \ln \frac{1}{\tilde{V}}, & \tilde{V} \ll 1 \\ \frac{1}{\tilde{V}^2} \left[1 + \sqrt{\frac{\pi}{2\tilde{V}}} \sin\left(2\tilde{V} - \frac{\pi}{4}\right) \right], & \tilde{V} \gg 1 \end{cases}$$

where $\tilde{V} = 2LeV/\hbar v_{2D}$. The conductance exhibits oscillations as a function of the bias voltage due to the geometrical quantization.

Consider now the energies $eV > \Delta \gg \Gamma$. Not very close to the gap edge Δ the Andreev reflection is small, while

$$|r_N|^2 = \exp[-2\text{Im}(\delta_+ - \delta_-)] = \exp\left[-\frac{4L\Gamma}{\hbar v_x} \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}\right].$$

For short contacts $L\Gamma/\hbar v_{2D} \ll 1$ we find

$$R_0 \frac{dI}{dV} = \frac{2\pi L\Gamma}{\hbar v_{2D}} \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}. \quad (51)$$

Region close to the gap edge, $|\epsilon - \Delta| \rightarrow 0$ needs a special consideration. Here $u^2 \rightarrow v^2$, so that the normal reflection vanishes while the Andreev reflection grows up to $|r_A|^2 = 1$, and the conductance has a sharp peak as in the case of long contacts considered in Sec. IV A. The differential conductance is plotted in Fig. 7.

As was mentioned in Sec. III B, short leads are equivalent to the limit of low transmission tunnel contact. The differential conductance Eq. (51) at voltages $eV > \Delta$ is proportional to the superconducting density of states, as it is usually the case for tunnel contacts. However, in addition to the peak at the superconducting gap edge, the differential conductance has a low-energy peak which characterizes the induced superconducting gap ϵ_g in the 2D layer. In the voltage interval $\epsilon_g < eV < \Delta$, the conductance oscillates as a function of voltage due to the geometrical quantization in the regions where the leads overlap with the 2D layer. The contacts with long leads, $L\Gamma/\hbar v_{2D} \gg 1$, for $\epsilon > \Delta$ have an exponentially small normal reflection due to almost complete escape of quasiparticles into the superconductors, and thus their conductance coincides with that obtained in Sec. IV A.

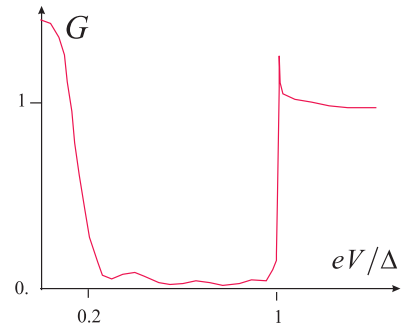


FIG. 7: Differential conductance $G = R_0(dI/dV)$ as a function of $V/100\Delta$. Here $\Gamma/\Delta = 0.1$ and $2L\Delta/\hbar v_{2D} = 10$.

V. DISCUSSION

We formulate the approach which can be used for spatially inhomogeneous and/or time-dependent problems associated with the induced superconductivity in low dimensional electronic systems including a 2D electron gas, graphene layer, etc. This approach is based on the so-called Fano–Anderson model which describes the decay of a resonance state coupled to a continuum¹⁴. We consider a 2D system placed in a contact with a bulk superconductor and use the tunnel approximation for coupling between the superconductor and the 2D electron layer in a way similar to that used in Refs.^{12,13} for impurities in a superconductor. We consider two particular examples of junctions made of a ballistic 2D electron gas placed under the superconducting electrodes of a finite length. For the case of a short symmetric S/2D/S junction we find the bound states localized in the junction and calculate the supercurrent as a function of the lead length. Next we consider the 2D/S junction and find

the IV curve for various lead lengths. We show that the differential conductance as a function of the bias voltage exhibits peaks which correspond to the induced and bulk superconducting gaps. The differential conductance also shows oscillations as a function of voltage due to geometrical quantization in the regions with the induced superconductivity.

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